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Efficient computational approaches for fractional-order Degasperis-Procesi and Camassa–Holm equations

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ABSTRACT

In this study, we present a comprehensive comparison of two powerful analytical techniques, Aboodh Adomian decomposition method (AADM) and homotopy perturbation transform method (HPTM), for obtaining series solutions of nonlinear partial differential equations, specifically focusing on Camassa–Holm (CH) and Degasperis–Procesi (DP) equations. These equations are widely used to describe various nonlinear wave phenomena in fluid mechanics, optical fibers, and other applications. By applying both AADM and HPTM to CH and DP equations, we demonstrate the effectiveness and efficiency of each method in terms of accuracy, convergence, and computational complexity. Furthermore, we provide a detailed analysis of the series solutions obtained by each method and discuss their respective advantages and limitations. The results reveal that both methods are capable of providing accurate and convergent series solutions for the considered equations. However, AADM shows a slightly better performance in terms of convergence rate and ease of implementation, making it a preferable choice for solving CH and DP equations. This comparative study serves as a useful reference for researchers and practitioners working in the field of nonlinear partial differential equations and their applications.

Introduction

The development of fractional calculus dates back to the 17th century, with the works of mathematicians such as Leibniz, Euler, and Laplace. However, it was not until the mid-20th century that fractional differential equations (FDEs) began to be used in scientific research. Since then, many researchers have studied FDEs and their applications, leading to the development of various methods and techniques for solving them [1–5]. Fractional differential equations are a powerful tool for modeling complex phenomena that cannot be adequately described by ordinary differential equations [6–10]. They are a generalization of classical calculus, allowing for the use of fractional derivatives instead of integer derivatives. The use of fractional derivatives opens up a new world of possibilities, allowing for the modeling of phenomena such as anomalous diffusion, viscoelasticity, and electrical conduction in complex media. This field of study has grown rapidly over the last few decades, with applications in physics, engineering, biology, and finance, among others. In this article, we will explore the basic concepts of fractional calculus and fractional differential equations, their properties, and some of their applications [11–14].

The study of fractional calculus has gained significant attention in recent years due to its potential applications in various fields of science and engineering. Several researchers have explored efficient and reliable techniques for solving fractional-order physical models and partial differential equations. In this regard, the works of Shah et al. have contributed significantly to the development of novel approaches for analyzing various nonlinear wave phenomena. For instance, in

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their recent publications, they have investigated fractional nonlinear regularized long-wave models, analyzed the behavior of optical solitons for nonlinear Schrodinger equations, and compared different analytical approaches for systems of time fractional partial differential equations. Furthermore, their work on fractional Kaup-Kupershmidt equations and Korteweg-De-Vries-type equations under the Atangana-Baleanu-Caputo operator has provided insights into the modeling of nonlinear waves in plasma and fluid [15–17]. This paper aims to review some of their recent contributions to the field of fractional calculus and highlight the significance of their findings. Specifically, we will discuss their works on fractional-order physical models involving ρ-Laplace transform, fractional partial differential equations, and nonlinear Boussinesq equation under the Atangana-Baleanu-Caputo operator [18–23].

The Camassa–Holm and Degasperis-Procesi equations are two important nonlinear partial differential equations that have attracted a lot of attention in the field of applied mathematics and theoretical physics. The Camassa–Holm equation was first introduced in 1993 by R. Camassa and D. Holm to describe the evolution of shallow water waves in the presence of dissipation and dispersion effects. On the other hand, the Degasperis-Procesi equation was proposed by A. Degasperis and M. Procesi in 1999 as a modification of the Camassa–Holm equation to account for the higher order nonlinearities in the wave propagation.

The solutions of these equations exhibit several interesting properties, such as the existence of solitons and other localized structures, the occurrence of wave breaking and rogue waves, and the formation of complex patterns in the wave profiles. These phenomena have important implications in many areas of science and engineering, including oceanography, fluid mechanics, optics, and nonlinear optics. In this review, we will provide a brief overview of the Camassa–Holm and Degasperis-Procesi equations, their mathematical properties, and some of their recent applications in different fields. We will also discuss some of the numerical methods and analytical techniques that have been used to study these equations and their solutions. Finally, we will highlight some of the open problems and challenges that still remain in this area of research. In this work, a modified β-equation, which has the following form, a family of important physical equations, is taken into consideration. [24]:

$$\frac{\partial^{\varsigma}\zeta(\beta,\upsilon)}{\partial\upsilon^{\varsigma}}-\frac{\partial}{\partial\upsilon}\left(\frac{\partial^{2}\zeta(\beta,\upsilon)}{\partial\beta^{2}}\right)+(\beta+1)\zeta^{2}(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}-\beta\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\left(\frac{\partial^{2}\zeta(\beta,\upsilon)}{\partial\beta^{2}}\right)-\zeta(\beta,\upsilon)\frac{\partial^{3}\zeta(\beta,\upsilon)}{\partial\beta^{3}}=0. \tag{1}$$

By selecting a value of $\beta=3$, the mDP model is produced.

$$\frac{\partial^{\varsigma}\zeta(\beta,\upsilon)}{\partial\upsilon^{\varsigma}}-\frac{\partial}{\partial\upsilon}\left(\frac{\partial^{2}\zeta(\beta,\upsilon)}{\partial\beta^{2}}\right)+4\zeta^{2}(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}-3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\left(\frac{\partial^{2}\zeta(\beta,\upsilon)}{\partial\beta^{2}}\right)-\zeta(\beta,\upsilon)\frac{\partial^{3}\zeta(\beta,\upsilon)}{\partial\beta^{3}}=0. \tag{2}$$

By selecting a value of 2 for β in Eq. (1), the result is the manifestation of the mCH model.

$$\frac{\partial^{\varsigma}\zeta(\beta,\upsilon)}{\partial\upsilon^{\varsigma}}-\frac{\partial}{\partial\upsilon}\left(\frac{\partial^{2}\zeta(\beta,\upsilon)}{\partial\beta^{2}}\right)+3\zeta^{2}(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}-2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\left(\frac{\partial^{2}\zeta(\beta,\upsilon)}{\partial\beta^{2}}\right)-\zeta(\beta,\upsilon)\frac{\partial^{3}\zeta(\beta,\upsilon)}{\partial\beta^{3}}=0. \tag{3}$$

In this study, a novel concept is proposed for solving the fractional mDP and mCH equations using the Caputo operator. The mCH and mDP models are similar to the incompressible Euler equation and have been found to be fully integrable with a Lax pair. This model has been the subject of various numerical studies, with Liu and Ouyang [25] developing new solitary wave solutions, Dubey et al. [26] proposing a q-homotopy analysis approach, Behera and Mehra [27] using a wavelet-optimized finite difference approach, Kader and Latif [28] employing a Lie symmetry technique, and Yousif et al. [29] developing two techniques, the homotopy perturbation method and variational iteration method.

One of the earliest studies of the fractional DPE (FDPE) was carried out by Abdeljawad and Baleanu (2014), who derived a generalized version of the FDPE and investigated its properties [30]. A similar approach was taken by Wang et al. (2015), who considered the fractional CHE (FCHE) and showed that it could be used to model shallow water waves with fractional dissipation [31]. More recently, Abdeljawad et al. (2019) studied the properties of the fractional-order DPE and CHE using the Adomian decomposition method [32]. Another interesting approach to studying the fractional DPE and CHE was taken by Li et al. (2020), who considered the fractional Fourier transform (FrFT) and its application to solving fractional partial differential equations [33]. Wang et al. (2017) used the homotopy perturbation method to study the fractional CHE and showed that it could be used to model shallow water waves with fractional dissipation [34]. Abdeljawad et al. (2017) considered the fractional-order DPE and CHE with Riesz-Feller derivative, and showed that the solutions could be expressed in terms of the Riesz-Feller fractional integral [35]. In a different approach, Yang and Wei (2020) used the variational iteration method to study the fractional CHE with Riemann–Liouville derivative, and obtained an analytical solution for the equation [36]. Finally, it is worth noting that some researchers have also investigated the numerical methods for solving fractional DPE and CHE equations. For example, Wu et al. (2017) used the finite difference method to solve the fractional DPE, and showed that the method was effective in predicting the behavior of the solution [37]. Similarly, Li et al. (2018) used the spectral method to solve the fractional CHE, and showed that the method was accurate and efficient [38]. Overall, the study of fractional-order DPE and CHE equations is an active and growing area of research, with many recent studies exploring the properties and behavior of these equations using various analytical and numerical techniques. These studies suggest that fractional calculus techniques can provide a powerful and flexible approach to modeling complex physical phenomena.

Homotopy Perturbation Method (HPM) has gained increasing attention in the field of engineering and applied mathematics. Researchers have been exploring ways to enhance the accuracy and efficiency of this method in solving complex problems. One approach is the coupling of the HPM with other techniques, such as the Enhanced Perturbation Method (EPM), as demonstrated by Li XX and He CH in their paper published in the Journal of Low Frequency Noise, Vibration & Active Control in December 2019. Another example of this is the modified HPM proposed by N. Anjum et al. for the analysis of electrically actuated microbeams-based microelectromechanical systems, which was published in Facta Universitatis Series: Mechanical Engineering in 2021. In addition, Ji-Huan He and Yusry O. El-Dib presented the Enhanced Homotopy Perturbation Method (EHPM) for the axial vibration of strings in the same journal and year. These studies demonstrate the ongoing efforts to improve the HPM and its variants for a wide range of applications in engineering and science [39–41]. The field of fractional calculus has seen significant advancements in recent years, with researchers developing new methods to solve complex mathematical problems involving fractional derivatives. One such promising method is the Aboodh transformation-based homotopy perturbation method, which has been demonstrated to be effective in solving fractional calculus problems. In a recent article published in Frontiers in Physics, Huiqiang Tao, Naveed Anjum, and Yong-Ju Yang explored the potential of this method and provided evidence of its success in various applications. In this paper, we will discuss the Aboodh transformation-based homotopy perturbation method and its promising prospects for the future of fractional calculus [42].

In the field of mathematical analysis, the Adomian decomposition method (ADM) and the Homotopy perturbation method (HPM) are two popular techniques used to solve nonlinear differential equations. ADM was introduced by George Adomian in 1986, while HPM was developed by J.H. He in 1999. Both methods have been shown to be effective and efficient for solving a wide range of nonlinear problems in various fields such as engineering, physics, and finance. In recent years, the ZZ transformation has been introduced as a new tool to improve the performance of these two methods [43,44]. This transformation was first proposed by A. Yildirim and H. Kocak in 2011, and it has been applied to the ADM and HPM to enhance their convergence and accuracy [45]. The ZZ transformation is a type of power series expansion that transforms the original nonlinear differential equation into a linear one, making it easier to solve using the ADM or HPM. In this paper, we will review the ADM and HPM methods, and we will also discuss the application of the ZZ transformation in these methods. We will provide a detailed description of the methods, their advantages, and limitations, and we will present examples to demonstrate their effectiveness. Additionally, we will analyze the performance of the methods with and without the ZZ transformation, and we will compare their results with other existing methods. Overall, this paper aims to provide a comprehensive overview of the ADM, HPM, and ZZ transformation, and their applications in solving nonlinear differential equations.

Fundamental definitions

Definition 1. The Aboodh transform (AT) of a function $\theta(\beta)$ is given as [46,47]

$$C = \left\{\theta : |\theta(\beta)| < Be^{p_j|\beta|}, \text{ if } \beta \in (-1)^i \times [0,\infty), j = 1,2; \left(B, p_1, p_2 > 0\right)\right\}$$

can be defined as

$$\mathcal{A}[\theta(\beta)] = \mathcal{M}(\psi)$$

which is given by

$$\mathcal{A}[\theta(\beta)] = \frac{1}{\psi}\int_0^\infty \theta(\beta)e^{-\psi\beta}d\beta = \mathcal{M}(\psi), \quad p_1 \le \psi \le p_2$$

Definition 2. The inverse Aboodh transform of a function $\theta(\beta)$ is defined as [46,47]

$$\theta(\beta) = \mathcal{A}^{-1}[\mathcal{M}(\psi)].$$

Definition 3. The Mittag-Leffler function is a special term that often occurs naturally in the solution of fractional calculus is given as [46,47]

$$E_{\wp}(Z) = \sum_{\rho=0}^{\infty}\frac{Z^{\rho}}{\Gamma(\rho\wp+1)}, \quad \wp, Z \in \mathbb{C},$$

In generalized form, it is expressed as follow:

$$E^{\xi}_{\wp,\gamma} = \sum_{\rho=0}^{\infty}\frac{Z^{\rho}(\xi)_{\rho}}{\Gamma(\gamma+\rho\wp)\rho!}, \quad \wp, \gamma, Z \in \mathbb{C},$$

Definition 4. The fractional AB derivative is a concept that pertains to the function $\theta \in H^1(0,1)$, where $0 < \wp < 1$. The definition of the fractional AB derivative is as follows [46,47]:

$${}^{ABC}_{\ 0}D^{\wp}_{\beta}\theta(\beta) = \frac{N(\wp)}{1-\wp}\int_0^{\beta}\theta'(x)E_{\wp}\left(\frac{-\wp(\beta-x)^{\wp}}{1-\wp}\right)dx$$

Definition 5. Let θ be an element in the Sobolev space $H^1(0,1)$ and let $0 < \omega < 1$, then the fractional AB derivatives can be defined using the Riemann–Liouville approach [46,47]

$${}^{ABR}_{\ 0}D^{\wp}_{\beta}\theta(\beta) = \frac{N(\wp)}{1-\wp}\frac{d}{d\beta}\int_0^{\beta}\theta(x)E_{\wp}\left(\frac{-\wp(\beta-x)^{\wp}}{1-\wp}\right)dx,$$

The normalization function $N(\wp)$ satisfies the requirement of being positive and has the values of $N(0) = 1$ and $N(1) = 1$.

Theorem 1. *The fractional AB operator of Laplace transformation in the presence of Caputo is given as [46,47]:*

$$\mathcal{L}\left[{}^{ABC}_{\ 0}D^{\wp}_{\beta}\theta(\beta)\right] = \frac{N(\wp)}{1-\wp}\times\frac{s^{\wp}F(s)-s^{\wp-1}f(0)}{s^{\wp}+\frac{\wp}{1-\wp}},$$

Additionally, the Laplace transformation of the fractional AB derivative when utilizing the Riemann–Liouville method is represented as

$$\mathcal{L}\left[{}^{ABR}_{\ 0}D^{\wp}_{\beta}\theta(\beta)\right] = \frac{N(\wp)}{1-\wp}\times\frac{s^{\wp}F(s)}{s\wp+\frac{\wp}{1-\wp}}.$$

Theorem 2. *The Aboodh transformation of a fractional AB operator in the presence of Caputo can be defined as follows: if $\mathcal{M}(\psi)$ represents the Aboodh transformation of $\theta(\beta) \in C$ and the Laplace transform of $\theta(\beta) \in C$ is $\theta(s)$ [46,47]*

$$\mathcal{M}\left({}^{ABC}_{\ 0}D^{\wp}_{\beta}\theta(\beta)\right) = \frac{N(\wp)\left(\mathcal{M}(\psi)-\psi^{-2}\theta(0)\right)}{1-\wp+\wp\psi^{-\wp}}.$$

Definition 6. The Aboodh transformation of a fractional AB operator in the context of Riemann–Liouville is defined as follows: Let $\mathcal{M}(\psi)$ represent the Aboodh transformation of $\theta(\beta)$ which is an element of C. Additionally, let $\theta(s)$ be the Laplace transform of $\theta(\beta) \in C$ [46,47]

$$\mathcal{M}\left({}^{ABR}_{\ 0}D^{\wp}_{\beta}\theta(\beta)\right) = \frac{N(\wp)\mathcal{M}(\psi)}{1-\wp+\wp\psi^{-\wp}}.$$

Procedure of HPTM

The HPTM method is presented as a solution for solving FPDEs.

$$D^{\wp}_{\upsilon}\zeta(\beta,\upsilon) = \mathcal{P}_1[\beta]\zeta(\beta,\upsilon) + \mathcal{R}_1[\beta]\zeta(\beta,\upsilon), \quad 0 < \wp \le 1, \tag{4}$$

with the initial condition

$$\zeta(\beta,0) = \xi(\beta).$$

By applying the AT method, we obtain that $D^{\wp}_{\upsilon} = \frac{\partial^{\wp}}{\partial\upsilon^{\wp}}$ is a Caputo operator of order $\wp$, and $\mathcal{P}_1[\beta]$ and $\mathcal{R}_1[\beta]$ are linear and nonlinear terms respectively.

$$A[D^{\wp}_{\upsilon}\zeta(\beta,\upsilon)] = A[\mathcal{P}_1[\beta]\zeta(\beta,\upsilon) + \mathcal{R}_1[\beta]\zeta(\beta,\upsilon)], \tag{5}$$

By utilizing the differential characteristic of AT, we obtain

$$M(u) = u\zeta(\beta,0) + \left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A[\mathcal{P}_1[\beta]\zeta(\beta,\upsilon) + \mathcal{R}_1[\beta]\zeta(\beta,\upsilon)]. \tag{6}$$

By utilizing the inverse AT method, we can obtain:

$$\zeta(\beta,\upsilon) = \zeta(\beta,0) + A^{-1}[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A[\mathcal{P}_1[\beta]\zeta(\beta,\upsilon) + \mathcal{R}_1[\beta]\zeta(\beta,\upsilon)]]. \tag{7}$$

By utilizing HPM on Eq. (12), we are able to improve its effectiveness.

$$\zeta(\beta,\upsilon) = \sum_{k=0}^{\infty}\epsilon^k\zeta_k(\beta,\upsilon). \tag{8}$$

where the homotopy parameter $\epsilon \in [0,1]$.

The non-linear component in Eq. (8) can be expressed as

$$\mathcal{R}_1[\beta]\zeta(\beta,\upsilon) = \sum_{k=0}^{\infty}\epsilon^k H_k(\zeta), \tag{9}$$

The method of obtaining polynomials is described as:

$$H_k(\zeta_0,\zeta_1,\ldots,\zeta_n) = \frac{1}{\Gamma(n+1)}D^k_{\epsilon}\left[\mathcal{R}_1\left(\sum_{k=0}^{\infty}\epsilon^i\zeta_i\right)\right]_{\epsilon=0}, \tag{10}$$

where $D^k_{\epsilon} = \frac{\partial^k}{\partial\epsilon^k}$.

By combining (14) and (15) with (12), we obtain

$$\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\beta,\upsilon) = \zeta(\beta,0) + \epsilon\times\left(A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\{\mathcal{P}_1\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\beta,\upsilon) + \sum_{k=0}^{\infty}\epsilon^k H_k(\zeta)\}\right]\right). \tag{11}$$

By analyzing the correlation of the coefficient of ϵ, we arrive at the conclusion that

$$\begin{aligned}
\epsilon^0 &: \zeta_0(\beta,\upsilon)=\zeta(\beta,0),\\
\epsilon^1 &: \zeta_1(\beta,\upsilon)\\
&= A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A(\mathcal{P}_1[\beta]\zeta_0(\beta,\upsilon)+H_0(\zeta))\right],\\
\epsilon^2 &: \zeta_2(\beta,\upsilon)\\
&= A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A(\mathcal{P}_1[\beta]\zeta_1(\beta,\upsilon)+H_1(\zeta))\right],\\
&.\\
&.\\
&.\\
\epsilon^k &: \zeta_k(\beta,\upsilon)\\
&= A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A(\mathcal{P}_1[\beta]\zeta_{k-1}(\beta,\upsilon)+H_{k-1}(\zeta))\right],
\end{aligned}$$

$$k>0, k\in N. \tag{12}$$

Therefore, the approximation in Eq. (8) takes the form of a series

$$\zeta(\beta,\upsilon)=\lim_{M\to\infty}\sum_{k=1}^{M}\zeta_k(\beta,\upsilon). \tag{13}$$

Procedure of ATDM

The following is a presentation of the ATDM method to resolve the FPDEs:

$$D_\upsilon^{\wp}\zeta(\beta,\upsilon)=\mathcal{P}_1(\beta,\upsilon)+\mathcal{R}_1(\beta,\upsilon), \qquad 0<\wp\le 1, \tag{14}$$

with the initial condition

$$\zeta(\beta,0)=\xi(\beta).$$

By performing the AT, the result is as follows:

The notation $D_\upsilon^{\wp}=\frac{\partial^{\wp}}{\partial\upsilon^{\wp}}$ represents the Caputo operator of order $\wp$. $\mathcal{P}_1$ and $\mathcal{R}_1$ refer to linear and non-linear functions, respecti

$$A[D_\upsilon^{\wp}\zeta(\beta,\upsilon)]=A[\mathcal{P}_1(\beta,\upsilon)+\mathcal{R}_1(\beta,\upsilon)], \tag{15}$$

By utilizing the differentiating attribute of AT, we obtain

$$M(u)=u\zeta(\beta,0)+\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A[\mathcal{P}_1(\beta,\upsilon)+\mathcal{R}_1(\beta,\upsilon)], \tag{16}$$

By utilizing the inverse AT method, we have the ability to

$$\zeta(\beta,\upsilon)=\zeta(\beta,0)+A^{-1}[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)]A[\mathcal{P}_1(\beta,\upsilon)+\mathcal{R}_1(\beta,\upsilon)]. \tag{17}$$

The solution for breaking down $\zeta(\beta,\upsilon)$ is:

$$\zeta(\beta,\upsilon)=\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon). \tag{18}$$

The nonlinear component in Eq. (19) can be depicted as

$$\mathcal{R}_1(\beta,\upsilon)=\sum_{m=0}^{\infty}\mathcal{A}_m(\zeta). \tag{19}$$

with

$$\mathcal{A}_m\left(\zeta_0,\zeta_1,\zeta_2,\dots,\zeta_m\right)=\frac{1}{m!}\left[\frac{\partial^m}{\partial\ell^m}\left\{\mathcal{R}_1\left(\sum_{m=0}^{\infty}\ell^m\zeta_m\right)\right\}\right]_{\ell=0}, \quad m=0,1,2,\dots \tag{20}$$

By combining sources (24) and (26) in (23), we have obtained

$$\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)=\zeta(\beta,0)+A^{-1}\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\times\left[A\left\{\mathcal{P}_1\left(\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)\right)+\sum_{m=0}^{\infty}\mathcal{A}_m(\zeta)\right\}\right]. \tag{21}$$

Thus we get

$$\zeta_0(\beta,\upsilon)=\zeta(\beta,0), \tag{22}$$

$$\zeta_1(\beta,\upsilon)=A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\{\mathcal{P}_1(\zeta_0)+\mathcal{A}_0\}\right],$$

In general, when m is greater than or equal to 1, we observe that

$$\zeta_{m+1}(\beta,\upsilon)=A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\{\mathcal{P}_1(\zeta_m)+\mathcal{A}_m\}\right].$$

Application

Example

Consider the fractional mDP equation is given as

$$\frac{\partial^{\wp}\zeta(\beta,\upsilon)}{\partial\upsilon^{\wp}}-\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}-3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}=0, \quad 0<\wp\le 1, \tag{23}$$

with the initial condition

$$\zeta(\beta,0)=-\frac{15}{8}\operatorname{sech}^2(\frac{\beta}{2})$$

Applying the AT, we get

$$A\left(\frac{\partial^{\wp}\zeta}{\partial\upsilon^{\wp}}\right)=A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right], \tag{24}$$

By utilizing the differentiating aspect of AT, the result can be obtained

$$M(u)=u\zeta(\beta,0)+\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]. \tag{25}$$

By utilizing the inverse AT method, we are able to

$$\begin{aligned}
\zeta(\beta,\upsilon)&=\zeta(\beta,0)+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right],\\
\zeta(\beta,\upsilon)&=(-\frac{15}{8}\operatorname{sech}^2(\frac{\beta}{2}))+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\times\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right].
\end{aligned} \tag{26}$$

With the use of the HPM method, we have been able to

$$\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\beta,\upsilon)=(-\frac{15}{8}\,\mathrm{sech}^2(\frac{\beta}{2}))+\left(A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\right.\right.$$
$$\times A\left[\left(\sum_{k=0}^{\infty}\epsilon^k\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta_k(\beta,\upsilon)}{\partial\beta^2})\right.\right.$$
$$\left.-4\sum_{k=0}^{\infty}\epsilon^k\zeta_k^2(\beta,\upsilon)\sum_{k=0}^{\infty}\epsilon^k\frac{\partial\zeta_k(\beta,\upsilon)}{\partial\beta}\right)+3\sum_{k=0}^{\infty}\epsilon^k\frac{\partial\zeta_k(\beta,\upsilon)}{\partial\beta}\sum_{k=0}^{\infty}\epsilon^k\frac{\partial^2\zeta_k(\beta,\upsilon)}{\partial\beta^2}$$
$$\left.\left.\left.+\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\beta,\upsilon)\sum_{k=0}^{\infty}\epsilon^k\frac{\partial^3\zeta_k(\beta,\upsilon)}{\partial\beta^3}\right]\right]\right). \tag{27}$$

By analyzing the relationship between the coefficient of ϵ, we can deduce that

$$\epsilon^0:\zeta_0(\beta,\upsilon)=-\frac{15}{8}\,\mathrm{sech}^2(\frac{\beta}{2}),$$
$$\epsilon^1:\zeta_1(\beta,\upsilon)=A^{-1}\left(\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta_0(\beta,\upsilon)}{\partial\beta^2})\right.\right.$$
$$-4\zeta_0^2(\beta,\upsilon)\frac{\partial\zeta_0(\beta,\upsilon)}{\partial\beta}$$
$$\left.\left.+3\frac{\partial\zeta_0(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta_0(\beta,\upsilon)}{\partial\beta^2})+\zeta_0(\beta,\upsilon)\frac{\partial^3\zeta_0(\beta,\upsilon)}{\partial\beta^3}\right]\right)$$
$$=-450\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right)\left(1-\wp+\frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right)$$
$$\vdots$$

In conclusion, the outcome of the series is presented as follows:

$$\zeta(\beta,\upsilon)=\zeta_0(\beta,\upsilon)+\zeta_1(\beta,\upsilon)+\cdots$$
$$\zeta(\beta,\upsilon)=-\frac{15}{8}\,\mathrm{sech}^2(\frac{\beta}{2})-450\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right)$$
$$\times\left(1-\wp+\frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right)+\cdots$$

Utilizing the ATDM

Applying the AT, we get

$$A\left\{\frac{\partial^{\wp}\zeta}{\partial\upsilon^{\wp}}\right\}=A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.$$
$$\left.+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right], \tag{28}$$

By utilizing the differentiating characteristic of AT, it is possible to obtain the desired result.

$$\frac{1}{\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)}\{M(u)-u\zeta(\beta,0)\}=A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.$$
$$\left.+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right], \tag{29}$$

$$M(u)=u\zeta(\beta,0)+\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.$$
$$\left.+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]. \tag{30}$$

Utilizing the inverse AT method, we possess

$$\zeta(\beta,\upsilon)=\zeta(\beta,0)+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.\right.\right.$$
$$-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}$$
$$\left.\left.\left.+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right],$$
$$\zeta(\beta,\upsilon)=(-\frac{15}{8}\,\mathrm{sech}^2(\frac{\beta}{2}))+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\right.$$
$$\times\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.\right.$$
$$\left.\left.\left.+3\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right]. \tag{31}$$

The approximation in the form of a series is expressed as

$$\zeta(\beta,\upsilon)=\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon) \tag{32}$$

with $\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}=\sum_{m=0}^{\infty}\mathcal{A}_m$, $\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})=\sum_{m=0}^{\infty}\mathcal{B}_m$ and $\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}=\sum_{m=0}^{\infty}\mathcal{C}_m$ are the Adomian polynomials which shows the nonlinear terms, and

$$\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)=\zeta(\beta,0)-A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.\right.\right.$$
$$\left.\left.\left.-4\sum_{m=0}^{\infty}\mathcal{A}_m+3\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m\right]\right\}\right],$$
$$\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)=(-\frac{15}{8}\,\mathrm{sech}^2(\frac{\beta}{2}))-A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\right.$$
$$\left.\times\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-4\sum_{m=0}^{\infty}\mathcal{A}_m+3\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m\right]\right\}\right]. \tag{33}$$

Comparing both sides leads to the derivation of the recursive algorithm:

$$\zeta_0(\beta,\upsilon)=-\frac{15}{8}\,\mathrm{sech}^2(\frac{\beta}{2}),$$

On $m=0$

$$\zeta_1(\beta,\upsilon)=-450\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right)\left(1-\wp+\frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right)$$

Lastly, the outcome in the form of a series is expressed as

$$\zeta(\beta,\upsilon)=\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)=\zeta_0(\beta,\upsilon)+\zeta_1(\beta,\upsilon)+\cdots$$
$$\zeta(\beta,\upsilon)=-\frac{15}{8}\,\mathrm{sech}^2(\frac{\beta}{2})-450\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right)$$
$$\times\left(1-\wp+\frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right)+\cdots$$

Therefore, when $\wp$ equals 1, the precise outcome is obtained

$$\zeta(\beta,\upsilon)=-\frac{15}{8}\left[\mathrm{sech}^2\,\frac{1}{2}\left(\beta-\frac{5}{2}\upsilon\right)\right]. \tag{34}$$

In Fig. 1, a comparison of the analytical and exact solutions at $\wp=1$ for the Degasperis-Procesi equation is presented. Subfigure (a) illustrates the analytical solution obtained using methods like AADM and HPM, while subfigure (b) displays the exact solution for Example 1. The close resemblance between these two plots indicates the effectiveness of the applied methods in approximating the exact solution. Fig. 2 showcases the comparison of exact and different fractional order analytical solutions for Example 1, where subfigures (a) and (b) represent

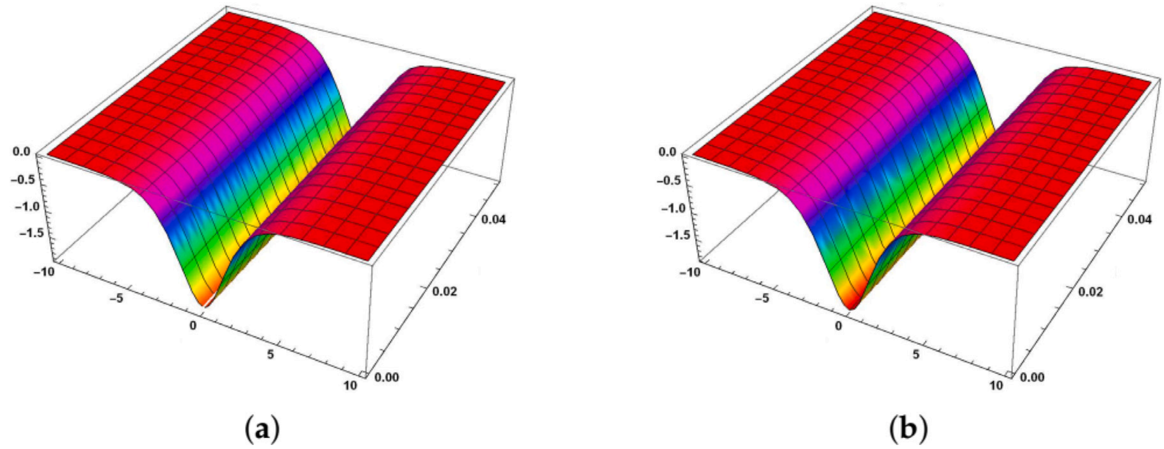

Fig. 1. Comparison of analytical and exact solutions at $\wp = 1$ (a) the analytical solution of the Degasperis-Procesi equation; (b) the exact solution of Example 1.

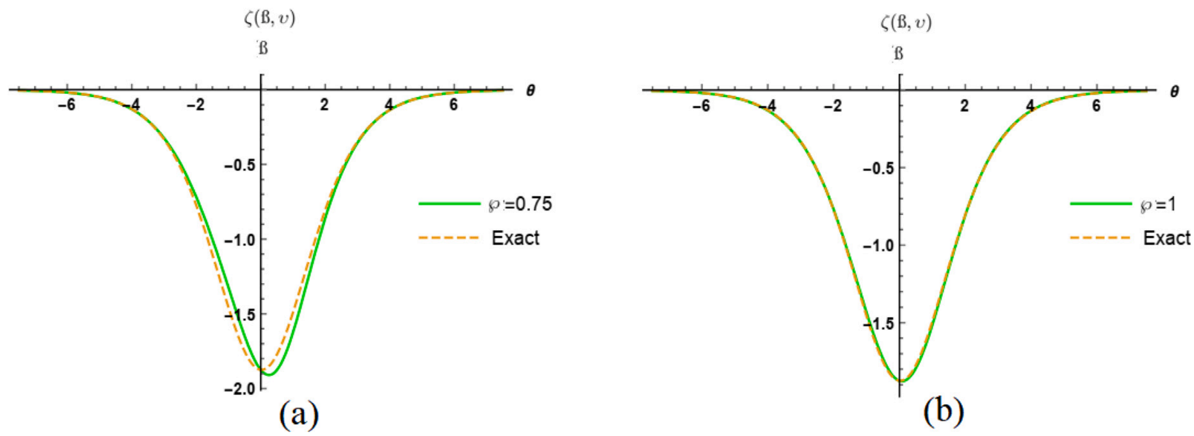

Fig. 2. Comparison of exact and different fractional order analytical solutions (a) $\wp = 0.75$ and (b) 1 of Example 1.

Table 1
Comparison of our methods and exact solution at $\wp = 1$ in addition with absolute error (AE).

$\upsilon = 0.01$	Exact solution	Our methods solution	AE of our methods
β	$\wp = 1$	$\wp = 1$	$\wp = 1$
1	−1.49154	−1.50142	2.3242749190E−03
2	−0.80253	−0.80532	3.8063761200E−04
3	−0.34656	−0.34432	3.5881918040E−04
4	−0.13570	−0.13432	2.5532137270E−04
5	−0.05110	−0.05070	1.1467922480E−04
6	−0.01896	−0.01872	4.5230501860E−05
7	−0.00699	−0.00690	1.7063410780E−05
8	−0.00257	−0.00255	6.3352901930E−06
9	−0.00094	−0.00089	2.3385053310E−06
10	−0.00034	−0.00034	8.6135638670E−07

the solutions at $\wp = 0.75$ and 1, respectively. The graphical comparison reveals the accuracy of the fractional order analytical solutions, demonstrating the adaptability and success of the applied methods for various fractional orders. Overall, these figures highlight the reliability and robustness of the AADM and HPM in providing precise solutions for nonlinear problems like the Degasperis-Procesi equation (see Table 1).

Example

Consider the fractional mCH equation is given as

$$\frac{\partial^{\wp}\zeta(\beta,\upsilon)}{\partial\upsilon^{\wp}} - \frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2}) + 3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta} - 2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})$$
$$-\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3} = 0, \quad 0 < \wp \le 1, \tag{35}$$

with the initial condition

$$\zeta(\beta,0) = -2\,\text{sech}^2(\frac{\beta}{2}).$$

Applying the AT, we get

$$A\left(\frac{\partial^{\wp}\zeta}{\partial\upsilon^{\wp}}\right) = A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2}) - 3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta} + 2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.$$
$$\left.+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]. \tag{36}$$

By utilizing the differential characteristic of AT, the result can be obtained

$$M(u) = u\zeta(\beta,0) + \left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2}) - 3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.$$
$$\left.+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2}) + \zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]. \tag{37}$$

By utilizing the inverse AT method, we have discovered a new approach for obtaining the desired outcome.

$$\zeta(\beta,\upsilon)=\zeta(\beta,0)+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.\right.\right.$$
$$-3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}$$
$$\left.\left.\left.+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right],$$
$$\zeta(\beta,\upsilon)=\left(-2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right)\right)$$
$$+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.\right.\right.$$
$$\left.\left.\left.+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right]. \tag{38}$$

Through the implementation of HPM procedure, we have achieved

$$\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\beta,\upsilon)=\left(-2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right)\right)+\left(A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\right.\right.$$
$$\times A\left[\left(\sum_{k=0}^{\infty}\epsilon^k\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta_k(\beta,\upsilon)}{\partial\beta^2})\right.\right.$$
$$\left.-3\sum_{k=0}^{\infty}\epsilon^k\zeta_k^2(\beta,\upsilon)\sum_{k=0}^{\infty}\epsilon^k\frac{\partial\zeta_k(\beta,\upsilon)}{\partial\beta}\right)+2\sum_{k=0}^{\infty}\epsilon^k\frac{\partial\zeta_k(\beta,\upsilon)}{\partial\beta}\sum_{k=0}^{\infty}\epsilon^k\frac{\partial^2\zeta_k(\beta,\upsilon)}{\partial\beta^2}$$
$$\left.\left.\left.+\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\beta,\upsilon)\sum_{k=0}^{\infty}\epsilon^k\frac{\partial^3\zeta_k(\beta,\upsilon)}{\partial\beta^3}\right]\right]\right). \tag{39}$$

By examining the relationship between the coefficient of ϵ and other factors, we can determine the correlation.

$$\epsilon^0:\zeta_0(\beta,\upsilon)=-2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right),$$
$$\epsilon^1:\zeta_1(\beta,\upsilon)=A^{-1}\left(\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta_0(\beta,\upsilon)}{\partial\beta^2})\right.\right.$$
$$-3\zeta_0^2(\beta,\upsilon)\frac{\partial\zeta_0(\beta,\upsilon)}{\partial\beta}$$
$$\left.\left.+2\frac{\partial\zeta_0(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta_0(\beta,\upsilon)}{\partial\beta^2})+\zeta_0(\beta,\upsilon)\frac{\partial^3\zeta_0(\beta,\upsilon)}{\partial\beta^3}\right]\right),$$
$$=-384\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right)\left(1-\wp+\frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right).$$
$$\vdots$$

Finally, the result of the series is presented as follows.

$$\zeta(\beta,\upsilon)=\zeta_0(\beta,\upsilon)+\zeta_1(\beta,\upsilon)+\cdots.$$
$$\zeta(\beta,\upsilon)=-2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right)-384\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right)$$
$$\times\left(1-\wp+\frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right)+\cdots.$$

Utilizing the ATDM

Using the AT, we get

$$A\left\{\frac{\partial^{\wp}\zeta}{\partial\upsilon^{\wp}}\right\}=A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.$$
$$\left.+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]. \tag{40}$$

By utilizing the distinct characteristic of AT, the result is obtained

$$\frac{1}{\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)}\{M(u)-u\zeta(\beta,0)\}=A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.$$
$$\left.+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right], \tag{41}$$

$$M(u)=u\zeta(\beta,0)+\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.$$
$$\left.+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]. \tag{42}$$

By utilizing inverse AT, we are able to

$$\zeta(\beta,\upsilon)=\zeta(\beta,0)+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.\right.\right.$$
$$-3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}$$
$$\left.\left.\left.+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right],$$
$$\zeta(\beta,\upsilon)=\left(-2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right)\right)+A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\right. \tag{43}$$
$$\times\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})-3\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}\right.\right.$$
$$\left.\left.\left.+2\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})+\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}\right]\right\}\right].$$

The series approximation is expressed as

$$\zeta(\beta,\upsilon)=\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon) \tag{44}$$

with $\zeta^2(\beta,\upsilon)\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}=\sum_{m=0}^{\infty}\mathcal{A}_m$, $\frac{\partial\zeta(\beta,\upsilon)}{\partial\beta}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})=\sum_{m=0}^{\infty}\mathcal{B}_m$ and $\zeta(\beta,\upsilon)\frac{\partial^3\zeta(\beta,\upsilon)}{\partial\beta^3}=\sum_{m=0}^{\infty}\mathcal{C}_m$ are the Adomian polynomials which shows the nonlinear terms, and

$$\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)=\zeta(\beta,0)-A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.\right.\right.$$
$$\left.\left.\left.-3\sum_{m=0}^{\infty}\mathcal{A}_m+2\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m\right]\right\}\right],$$
$$\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)=\left(-2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right)\right)-A^{-1}\left[\left(\frac{1-\wp+\wp u^{-\wp}}{N(\wp)}\right)\right.$$
$$\times\left\{A\left[\frac{\partial}{\partial\upsilon}(\frac{\partial^2\zeta(\beta,\upsilon)}{\partial\beta^2})\right.\right.$$
$$\left.\left.\left.-3\sum_{m=0}^{\infty}\mathcal{A}_m+2\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m\right]\right\}\right]. \tag{45}$$

The both sides comparisons of gives the recursive method:

$$\zeta_0(\beta,\upsilon)=-2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right).$$

On $m=0$

$$\zeta_1(\beta,\upsilon)=-384\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right)\left(1-\wp+\frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right).$$

Lastly, the series type solution is given as

$$\zeta(\beta,\upsilon)=\sum_{m=0}^{\infty}\zeta_m(\beta,\upsilon)=\zeta_0(\beta,\upsilon)+\zeta_1(\beta,\upsilon)+\cdots.$$

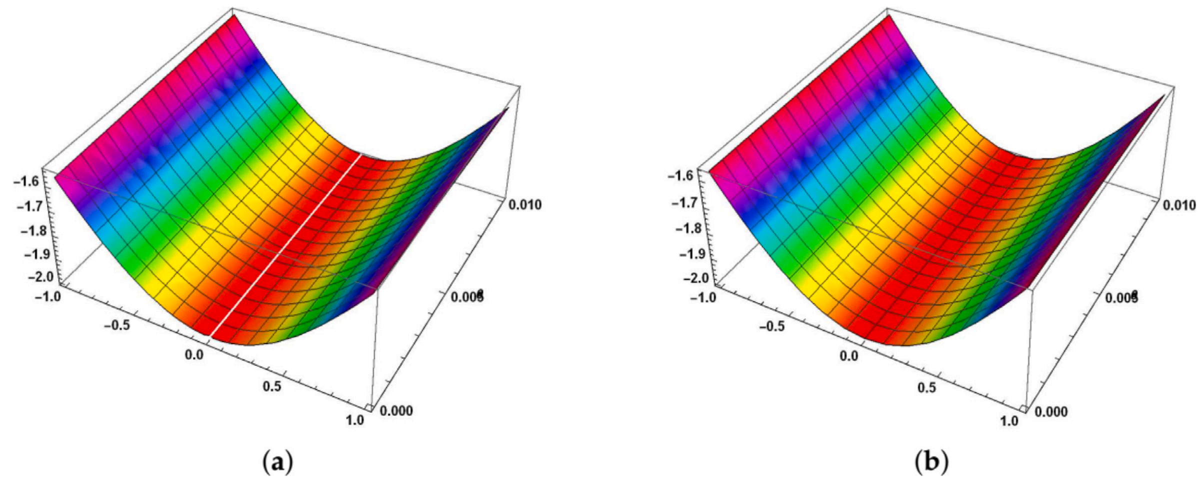

Fig. 3. Comparison of analytical and exact solutions at $\wp = 1$ (a) the analytical solution of the Camassa–Holm Equation; (b) the exact solution of Example 1.

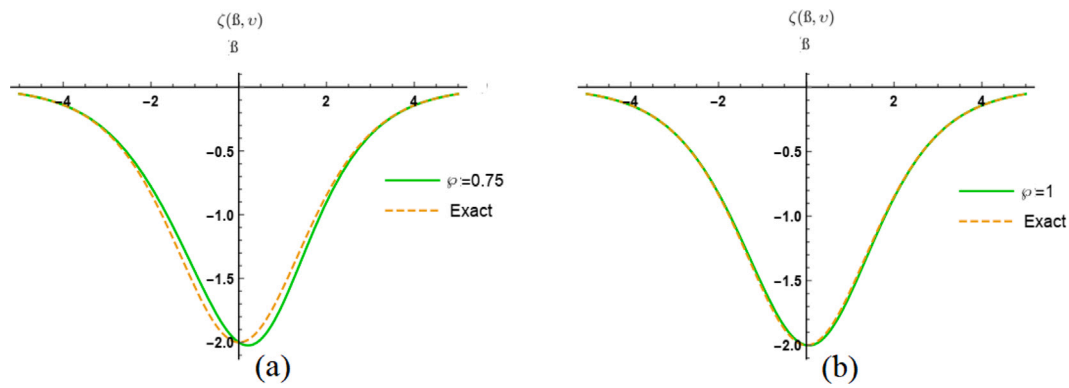

Fig. 4. Comparison of exact and different fractional order analytical solutions (a) $\wp = 0.75$ and (b) 1 of Example 1.

$$\zeta(\beta,\upsilon) = -2\,\mathrm{sech}^2\left(\frac{\beta}{2}\right) - 384\,\mathrm{csch}^5(\beta)\sinh^6\left(\frac{\beta}{2}\right) \times \left(1 - \wp + \frac{\wp\upsilon^{\wp}}{\Gamma(\wp+1)}\right) + \cdots.$$

The exact result is

$$\zeta(\beta,\upsilon) = -2\,\mathrm{sech}^2\left(\frac{\beta-\upsilon}{2}\right). \tag{46}$$

In Fig. 3, a comparison of the analytical and exact solutions at $\wp = 1$ for the Camassa–Holm Equation is presented. Subfigure (a) illustrates the analytical solution obtained using methods like AADM and HPM, while subfigure (b) displays the exact solution for Example 2. The close resemblance between these two plots indicates the effectiveness of the applied methods in approximating the exact solution. Fig. 4, showcases the comparison of exact and different fractional order analytical solutions for Example 2, where subfigures (a) and (b) represent the solutions at $\wp = 0.75$ and 1, respectively. The graphical comparison reveals the accuracy of the fractional order analytical solutions, demonstrating the adaptability and success of the applied methods for various fractional orders. Overall, these figures highlight the reliability and robustness of the AADM and HPM in providing precise solutions for nonlinear problems like the Degasperis-Procesi equation.

Conclusion

In conclusion, the Aboodh Adomian decomposition method (AADM) and Homotopy perturbation transform method (HPTM) have proven to be powerful and versatile techniques in finding series solutions to complex nonlinear problems. Through the application of these methods to the Camassa–Holm and Degasperis-Procesi equations, we have witnessed their efficacy in handling intricate scenarios in mathematical physics. The analytical solutions derived from these methods have provided valuable insights into the behaviors of nonlinear wave propagation, while also serving as useful tools in the prediction and understanding of various phenomena. Furthermore, these approaches have demonstrated their potential in extending our knowledge on other nonlinear partial differential equations. As the field of applied mathematics continues to evolve, the AADM and HPTM will undoubtedly remain essential tools in the study of nonlinear problems, contributing to advancements in both theoretical understanding and practical applications. In the future, researchers can extend the application of AADM and HPTM to a wider range of nonlinear partial differential equations, exploring new domains in applied mathematics and physics. Additionally, incorporating advanced numerical techniques and computational methods could enhance the accuracy and efficiency of these approaches. Lastly, interdisciplinary collaborations can reveal novel real-world applications, further highlighting the importance of these methods in solving complex nonlinear problems.

Declaration of competing interest

The authors have no conflict of interest.

Data availability

Data will be made available on request.

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